EMBEDDING OF HYPERCUBE INTO CYLINDER

WEIXING JI, QINGHUI LIU, GUIZHEN WANG, AND ZHUOJIA SHEN

ABSTRACT. Task mapping in modern high performance parallel computers can be modeled as a graph embedding problem, which simulates the mapping as embedding one graph into another and try to find the minimum wirelength for the mapping. Though embedding problems have been considered for several regular graphs, such as hypercubes into grids, binary trees into grids, et al, it is still an open problem for hypercubes into cylinders. In this paper, we consider the problem of embedding hypercubes into cylinders to minimize the wirelength. We obtain the exact wirelength formula of embedding hypercube Q^r into cylinder $C_{2^3} \times P_{2^{r-3}}$ with $r \geq 3$.

Graph embedding and Hypercube and Cylinder and Parallel computing

1. INTRODUCTION

On one hand, a parallel program can be modeled as a task graph, in which the vertices of the graph represent a computing task, and the edges represent the communications among different tasks. On the other hand, a massive parallel computer has a large number of processing nodes that are connected together with an interconnection network. One of the key problems of efficient execution of parallel programs on these computers is how to find an optimal mapping from computing tasks to processing nodes, so that the communication overhead could be reduced when the tasks are run in parallel. Without loss of generality, this problem can also be modeled as an graph embedding problem, since task mapping is actually try to find an optimal embedding of computing task graph into the interconnection graph (network) with minimum link congestion. Unfortunately, task mapping is proved as a NP hard problem and heuristics algorithms are usually used to find an approximate solution for a given application and interconnection.

Researchers have been working on graph embedding for years and proposed a number of solution for regular graphs, such as hypercubes into grids [7], binary trees into grids [8], honeycomb into hypercubes [9], grids into grids [10]. This paper introduce a new technique to estimate the wirelength of embedding hypercube Q^r into cylinder $C_{2^3} \times P_{2^{r-3}}$ with $r \geq 3$.

The rest of the paper is organized as follows. We show the existing work on graph embedding in Section 2. Section 3 introduces the gray embedding and some useful property, which is used afterwards in the wirelength calculation. Section 4 discusses some composite sets with Cartesian production structure. The wirelength calculation of hypercube into cylinder is given in section 5. Conclusion and future work appear in Section 6.

2. Problem definition

Let G and H be finite graphs with n vertices. $V(G)$ and $V(H)$ denote the vertex sets of G and H respectively. $E(G)$ and $E(H)$ denote the edge sets of G and H respectively. An embedding [4] f of G into H is defined as follows:

(i) f is a bijective map from $V(G)$ to $V(H)$;

(ii) f is a one-to-one map from $E(G)$ to $\{Path_f(u,v) \ : \ Path_f(u,v)$ is a path in H between $f(u)$ and $f(v)\}$.

The edge congestion of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(e)$ denote the number of edges (u,v) of G such that e is in the path $Path_f(u,v)$ between $f(u)$ and $f(v)$ in H, in other words,

$$EC_f(e) = |\{(u,v) \in E(G) \ : \ e \in P_f(u,v)\}|.$$

For any $S \subset E(H)$, define

$$EC_f(S) = \sum_{e \in S} EC_f(e).$$

The edge congestion of an embedding f of G into H is given by

$$EC_f(G, H) = \max_{e \in E(H)} EC_f(e).$$

The minimum edge congestion of G into H

$$EC(G, H) = \min_{f:G \to H} EC_f(G, H),$$

where the minimum is taken over all embeddings f of G into H.

The edge congestion problem of G into H is to find an embedding of G into H that induces minimum edge congestion $EC(G, H)$.

The wirelength of an embedding f of G into H is given by

$$WL_f(G, H) = \sum_{(u,v) \in E(G)} d_H(f(u), f(v)) = \sum_{e \in E(H)} EC_f(e).$$

where $d_H(f(u), f(v))$ denote the length of the path $Path_f(u, v)$ in H. The wirelength of G into H is defined as

$$WL(G, H) = \min_{f:G \to H} WL_f(G, H),$$

where the minimum is taken over all embeddings f of G into H.

The wirelength problem of G into H is to find an embedding of G into H that induces minimum wirelength $WL(G, H)$.

Manuel et al([6]) find that the maximal subgraph problem play an important role in solving wirelength problem. For a graph G and an integer m,

$$I_G(m) = \max_{A \subset V(G), \ |A|=m} |I_G(A)|,$$

where

$$I_G(A) = \{(u, v) \in E(G) \ : \ u, v \in V(G)\}.$$

A subset $A \subset V(G)$ is called optimal if $|I_G(A)| = I_G(|A|)$.

The following lemmas are proved in [7]. Note that a set of edges of H is said to be an edge cut of H, if the removal of these edges results in a disconnection of H.

Lemma 2.1 (Congestion Lemma). *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1, H_2 and let $G_1 = f^{-1}(H_1)$, $G_2 = f^{-1}(H_2)$. Also S satisfy the following condition,*

(i) For every edge $(a, b) \in G_i$, $i = 1, 2$, $Path_f(a, b)$ has no edges in S.

(ii) For every edge (a, b) in G with $a \in G_1$ and $b \in G_2$, $Path_f(a, b)$ has exactly one edge in S.

(iii) G_1 is optimal.

Then $EC_f(S)$ is minimum and $EC_f(S) = r|V(G_1)| - 2|E(G_1)|$.

Lemma 2.2 (Partition Lemma). *Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, \cdots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then*

$$WL_f(G, H) = \sum_{i=1}^{p} WL_f(S_i).$$

We will discuss the embedding of following graphs.

Q^r, the graph of the r-dimensional hypercube, has vertex-set $\{0, 1\}^r$, the r-fold Cartesian product of $\{0, 1\}$. Thus $|V(Q^r)| = 2^r$. Q^r has an edge between two vertices (r-tuple of 0s and 1s) if they differ in exactly one entry.

The 1-dimensional grid with $d \geq 2$ vertices is denoted as P_d. The 1-dimensional cycle with $d \geq 2$ vertices is denoted as C_d. The 2-dimensional grid is defined as $P_{d_1} \times P_{d_2}$, where $d_i \geq 2$ is an integer for each $i = 1, 2$. The cylinder $C_{d_1} \times P_{d_2}$, where $d_1 \geq 2$ and $d_2 \geq 1$, is a $P_{d_1} \times P_{d_2}$ grid with a wraparound edge in each column (see e.g., Figure 2). The torus $C_{d_1} \times C_{d_2}$, where $d_1, d_2 \geq 2$, is a $P_{d_1} \times P_{d_2}$ grid with a wraparound edge in each column and a wrapround edge in each row.

It is conjectured that the wirelength of embedding hypercube Q^r into cycle C_{2^r} is $3 \cdot 2^{2r-3} - 2^{r-1}$. It is called CT conjecture [4, 2, 3, 7]. It is also conjectured in [6] such that the wirelength of embedding hypercube Q^r into cylinder $C_{2^{r_1}} \times P_{2^{r_2}}$ with positive integers $r_1 + r_2 = r$ is

$$2^{r_1}(2^{2r_2-1} - 2^{r_2-1}) + 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1}),$$

and the wirelength of embedding hypercube Q^r into torus $C_{2^{r_1}} \times C_{2^{r_2}}$ with positive integers $r_1 + r_2 = r$ is

$$2^{r_1}(3 \times 2^{2r_2-3} - 2^{r_2-1}) + 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1}).$$

Manuel et al ([6]) verified the case of embedding Q^r into cylinder $C_{2^2} \times P_{2^{r-2}}$ for $r \geq 2$. We prove in this paper that

Theorem 2.3. *For any $r \geq 3$, the wirelength of embedding Q^r into cylinder $C_{2^3} \times P_{2^{r-3}}$ is*

$$2^{r_1}(2^{2r_2-1} - 2^{r_2-1}) + 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1}),$$

where $r_1 = 3$ and $r_2 = r - 3$.

Remark 1. Our argument for Theorem 2.3 also valid for embedding of hypercube Q^6 into torus $C_8 \times C_8$. So

$$WL(Q^6, C_8 \times C_8) = 2 \times 8 \times 20 = 320.$$

3. Gray Embedding

Grid embedding plays an important role in computer architecture, and researchers believe that gray embedding minimize wirelength of emdedding hypercube into cycles, cylinders and torus [2, 6].

To construct gray embedding, we give first the bijection of $V(Q^r)$ to $V(P_{2^{r_1}} \times P_{2^{r_2}})$.

Given $d > 0$ and under Gray code list of d bits, every code corresponding to a number, e.g.,

$$g_d(0^d) = 0, \ g_d(0^{d-1}1) = 1, \ g_d(0^{d-2}11) = 2, \cdots, \ g_d(10^{r-1}) = 2^d - 1.$$

Define an embedding *gray* from Q^r into $C_{2^{r_1}} \times P_{2^{r_2}}$ with $r_1, r_2 \geq 2$ and $r_1 + r_2 = r$. The vertices of $C_{2^{r_1}} \times P_{2^{r_2}}$ have coordinates of the form (i, j), for $i = 0, 1, \cdots, 2^{r_1} - 1$, $j = 0, 1, \cdots, 2^{r_2} - 1$. Every vertex of Q^r correspond to a string in $\{0, 1\}^r$. Take any $w \in \{0, 1\}^r$, write $w = uv$, where $u \in \{0, 1\}^{r_1}$, $v \in \{0, 1\}^{r_2}$, define

$$gray(w) = (g_{r_1}(u), g_{r_2}(v)),$$

which corresponding to a unique vertex in $V(P_{2^{r_1}} \times P_{2^{r_2}})$.

As an example, we illustrate the embedding from $V(Q^4)$ to $V(P_4 \times P_4)$. By the above construction

$$g_2(00) = 0, \ g_2(01) = 1, \ g_2(11) = 2, \ g_2(10) = 3.$$

So we can directly get the map *gray* as shown in Figure 1.

Lemma 3.1. *Fix $r \geq 2$ and $r_1, r_2 \geq 1$ with $r_1 + r_2 = r$. Define a bijection gray from $V(Q^r)$ to $V(P_{2^{r_1}} \times P_{2^{r_2}})$ as above. For any edge (x, y) in hypercube Q^r, gray(x) and gray(y) are in the same row or the same column of $P_{2^{r_1}} \times P_{2^{r_2}}$.*
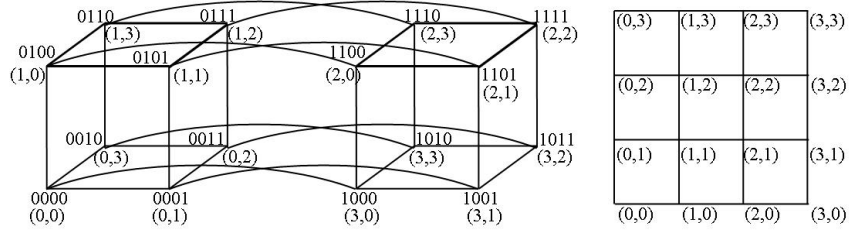
FIGURE 1.   The bijective map $gray: V(Q^4) \to V(P_4 \times P_4)$

Proof. Without loose of generality, let $x, y \in \{0,1\}^r$. Write $x = u_1 v_1$, $y = u_2 v_2$, with $|u_1| = |u_2| = r_1$ and $|v_1| = |v_2| = r_2$. Since x and y are different only in one bit, either $u_1 = u_2$ or $v_1 = v_2$. Hence either the x-coordinate or the y-coordinate of $gray(x)$ and $gray(y)$ are equal. The result of the lemma follows. $\square$ $\square$

Let $G = Q^r$ and $H = C_{2^{r_1}} \times P_{2^{r_2}}$ (respectively $H = C_{2^{r_1}} \times C_{2^{r_2}}$). Now we can construct graph embedding from G to H. We have already define a vertices bijection $gray$ from G to H. For any edge (x, y) in G, define $Path_{gray}(x, y)$ be the path in H between $gray(x)$ and $gray(y)$ with minimal number of edges.

4. STRUCTURE OF A CLASS OF COMPOSITE SETS

In our paper we will discuss some composite sets with Cartesian product structure.

A c-subcube of the r-cube is the subgraph of Q^r induced by the set of all vertices having the same value in some $r - c$ coordinates.

For any $0 < k < 2^r$, write

$$k = \sum_{i=1}^{m} 2^{c_i}, \quad 0 \le c_1 < c_2 < \cdots < c_m.$$

If S is a subgraph of Q^r which is a disjoint union of c_i-subcubes, $1 \le i \le m$, such that each c_j-subcube lies in a neighborhood of every c_i-subcube for any $j > i$, then S is called a composite set (or cubal, see [1, 5]).

Let S be a subgraph of Q^r with $|V(S)| = k > 0$. It is proved in [5] that S is a composite set if and only if it is optimal, or equivalently,

$$I_{Q^r}(S) = I_{Q^r}(k).$$

For any graph S and T, recall that the Cartesian product $S \times T$ of S, T is, $V(S \times T) = V(S) \times V(T)$, and $((u_1, u_2), (v_1, v_2)) \in E(S \times T)$ if and only if

$$(u_1, v_1) \in E(S),\ u_2 = v_2 \in V(T) \quad \text{or} \quad u_1 = v_1 \in V(S),\ (u_2, v_2) \in E(T).$$

Take any $r = r_1 + r_2$ with $r_1, r_2 \geq 1$. By definition of Cartesian product of graphs, it is direct to know that $Q^r = Q^{r_1} \times Q^{r_2}$. If S is a d_1-subcube of Q^{r_1}, T is a d_2-subcube of Q^{r_2}, then $S \times T$ is a $(d_1 + d_2)$-subcube of Q^r. Hence by definition of composite set, we see that if S is a subcube of Q^{r_1}, T is a composite set of Q^{r_2}, then $S \times T$ is a composite set of Q^r. So we get the following lemma

Lemma 4.1. *Let S and T are composite sets of Q^{r_1} and Q^{r_2} respectively. Suppose further that at least one of S and T is a subcube, then $S \times T$ is a composite set of $Q^{r_1} \times Q^{r_2}$.*

Recall that the (binary-reflected) Gray code list for d bits can be generated recursively from the list for $d - 1$ bits by reflecting the list (i.e. listing the entries in reverse order), concatenating the original list with the reversed list, prefixing the entries in the original list with a binary 0, and then prefixing the entries in the reflected list with a binary 1.

By this kind of recursive structure, we see that if $0 \leq k < 2^d$, then $g_d^{-1}(0 : k)$ is a composite set, where we use the matlab notation $m : n$ for any $0 \leq m \leq n$ to indicate the set $\{m, m+1, \cdots, n\}$ of consecutive integers.

In fact, write

$$k = \sum_{i=1}^m 2^{c_i}, \quad 0 \leq c_1 < c_2 < \cdots < c_m.$$

Let $a_{m+1} = 0$, and for $1 \leq i \leq m$,

$$a_i = \sum_{k=i}^m 2^{c_k}.$$

Then for any $1 \leq i \leq m$, $g_d^{-1}(a_{i+1} : a_i - 1)$ is a c_i-subcube in Q^d. These subcubes are disjoint, and each c_j-subcube lies in a neighborhood of every c_i-subcube for any $j > i$. So we get the following lemma

Lemma 4.2. *For any $d > 0$ and $0 \leq j < 2^d$, $g_d^{-1}(0 : j)$ is a composite set of hypercube Q^d.*

5. Hypercube into Cylinder

Now we can prove Theorem 2.3 by computing $WL(G, H)$ for $G = Q^r$, $H = C_8 \times P_{2^{r-3}}$. Denote $r_1 = 3$ and $r_2 = r - 3$.

To apply Congestion Lemma, we need to construct suitable edge cuts to form a partition. For $j = 1, 2, \cdots, 2^{r_2} - 1$, define edge cut

$$B_j = \{((i, j-1), (i, j)) \ : \ i = 0, 1, \cdots, 7\}.$$

Define edge cuts

$$
\begin{aligned}
A_1 &= \{((0, j), (1, j)), ((3, j), (4, j)) \ : \ j = 0, 1, \cdots, 2^{r_2} - 1\} \\
A_2 &= \{((1, j), (2, j)), ((6, j), (7, j)) \ : \ j = 0, 1, \cdots, 2^{r_2} - 1\} \\
A_3 &= \{((2, j), (3, j)), ((5, j), (6, j)) \ : \ j = 0, 1, \cdots, 2^{r_2} - 1\} \\
A_4 &= \{((4, j), (5, j)), ((7, j), (0, j)) \ : \ j = 0, 1, \cdots, 2^{r_2} - 1\}
\end{aligned}
$$

For any $1 \leq i \leq 4$, A_i disconnects H into two components X_i and X_i'. For any $0 \leq j < 2^{r_2} - 1$, B_j disconnects H into two components Y_j and Y_j'. We illustrate the case $r = 6$, $r_1 = r_2 = 3$ in Figure 2.
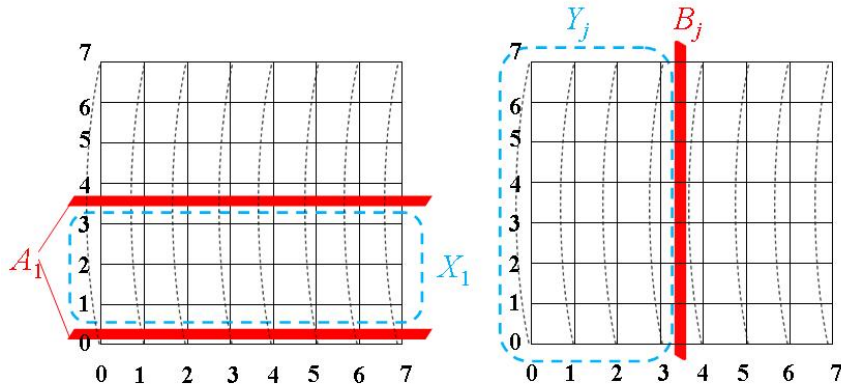

FIGURE 2.   The edge cut A_1 and B_j in $H = C_{2^3} \times P_{2^{r-3}}$

We discuss first A_1.

Let G_1 and G'_1 be the inverse images of X_1 and X'_1 under $gray$ respectively. By Lemma 3.1, the edge cut A_1 satisfies condition (ii) of the Congestion Lemma.

Now we show that A_1 satisfies condition (i). Since for any edge $(u, v) \in G$, $Path_{gray}(u, v)$ is the shortest path connecting $gray(u)$ and $gray(v)$, we only need to check whether there is a path from $(0, j)$ to $(4, j)$ for any $j = 0, 1, \cdots, 2^{r-3} - 1$. Notice that, if the Hamming distance of two codes is odd, then the gray coding distance is odd, and vice versa. This implies that the Hamming distance between $g_3^{-1}(0)$ and $g_3^{-1}(4)$ is even. So there are no edge path connecting $(0, j)$ and $(4, j)$.

Notice that $V(X_1) = \{1, 2, 3\} \times \{0, 1, \cdots, 2^{r_2} - 1\}$ for some k. It is direct to see that $g_3^{-1}(\{1, 2, 3\})$ is a composite set, and $g_{r_2}^{-1}\{0, 1, \cdots, 2^{r_2} - 1\}$ is a r_2-subcube. This implies that the subgraph G_1 is Cartesian product of a composite set and a subcube. By Lemma 4.1, G_1 is a composite set, and hence optimal. Thus by the Congestion Lemma, $EC_{gray}(A_1)$ is minimum.

The argument for A_2, A_3 and A_4 are analogous. For A_2, we see that $g_3^{-1}(\{2, 3, 4, 5, 6\})$ is a composite set, and there are no edge path connecting $(2, j)$ and $(6, j)$. For A_3, we see that $g_3^{-1}(\{3, 4, 5\})$ is a composite set, and there are no edge path connecting $(2, j)$ and $(6, j)$. For A_4, we see that $g_3^{-1}(\{5, 6, 7\})$ is a composite set, and there are no edge path connecting $(0, j)$ and $(4, j)$. Thus by the Congestion Lemma, $EC_{gray}(A_2)$, $EC_{gray}(A_3)$ and $EC_{gray}(A_4)$ are also minimum.

Fix any $1 \le j \le 2^{r_2} - 1$. Let G_j and G'_j be the inverse images of Y_j and Y'_j under $gray$ respectively. The edge cut B_j satisfies conditions (i) and (ii) of the Congestion Lemma. Note that $V(Y_j) = \{0, 1, \cdots, 7\} \times \{0, 1, \cdots, j - 1\}$. It is direct to see that $g_3^{-1}(0 : 7)$ is a 3-subcube. And by Lemma 4.2, $g_{r_2}^{-1}(0 : j)$ is a composite set. This implies that G_j is Cartesian product of a sub-hypercube of order 3 with a composite set. By Lemma 4.1, G_j is also a composite set, and hence is optimal. Thus by the Congestion Lemma, $EC_{gray}(B_j)$ is minimum.

The Partition lemma implies that $WL_{gray}(G, H)$ is minimum.

It is direct to compute that for any $r \geq 3$

$$WL_{gray}(Q^r, C_{2^3} \times P_{2^{r-3}}) = 2^{r_1}(2^{2r_2-1} - 2^{r_2-1}) + 2^{r_2}(3 \cdot 2^{2r_1-3} - 2^{r_1-1}),$$

where $r_1 = 3$ and $r_2 = r - 3$.

This proves Theorem 2.3.

6. Conclusion

Manuel et al get the exact wirelength of embedding of hypercube Q^r into cylinder $C_{2^2} \times P_{2^{r-3}}$ with $r \geq 2$. We prove in this paper that gray embedding minimizes the wirelength of embedding hypercube Q^r into cylinder $C_{2^3} \times P_{2^{r-3}}$ with $r \geq 3$, and hence get the exact wirelength of this case. We also get the exact wirelength of embedding hypercube Q^6 into torus $C_{2^3} \times C_{2^3}$.

We ever apply this method to study the case of embedding hypercube Q^r into cylinder $C_{2^4} \times P_{2^{r-4}}$ for any $r \geq 4$. We tried many embeddings(including gray embedding), but we can't get a partition to apply Congestion Lemma.

References

[1] A.J. Boals, A.K. Gupta, N.A. Sherwani, Incomplete hypercubes: Algorithms and embeddings, The Journal of Supercomputing, 8, 263-294(1994).

[2] J.D. Chavez, R. Trapp, The cyclic cutwidth of trees, Discrete Applied Mathematics, 87, 25-32(1998).

[3] J. Erbele, J.D. Chavez, R. Trapp, The Cyclic Cutwidth of Q_n, Manuscript, California State University, San Bernardino USA, 2003.

[4] C.-J. Guu, The circular wirelength problem for hypercubes, PhD dissertation, University of California, Riverside, 1997.

[5] L.H. Harper, Global Methods for Combinatorial Isoperimetric Problems, Cambridge University Press, 2004.

[6] P. Manuel, M. Arockiaraj, I. Rajasingh and B. Rajan, Embedding hypercubes into cylinders, snakes and caterpillars for minimizing wirelength, Discrete Applied Mathematics, 159(17), 2109-2116(2011).

[7] P. Manuel, I. Rajasingh, B. Rajan and H. Mercy, Exact wirelength of hypercube on a grid, Discrete Applied Mathematics, 157(7), 1486-1495(2009).

[8] J. Opatrny, D. Sotteau, Embeddings of complete binary trees into grids and extended grids with total vertex-congestion 1, Discrete Applied Mathematics 98, 237-254(2000).

[9] B. Doina, W.B. Wolfgang, B. Natasa, L. Shahram, An optimal embedding of honeycomb networks into hypercubes, Parallel Processing Letters, 14, 367-375(2004).

[10] M. Rottger, U.P. Schroeder, Efficient embeddings of grids into grids, Discrete Applied Mathematics, 108 (1-2) 143-173(2001).

DEPT. COMPUTER SCIENCE, BEIJING INSTITUTE OF TECHNOLOGY, BEIJING 100081, PR CHINA. QHLIU@BIT.EDU.CN, JWX@BIT.EDU.CN, GZWANG@BIT.EDU.CN, CHAOSDEFINITION@HOTMAIL.COM.